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Non-perturbative vacua and particle physics in M-theory

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Abstract: In this letter, we introduce a general theory for the construction of particle physics theories, with three families and realistic gauge groups, within the context of heterotic M-theory. This is achieved using semi-stable holomorphic gauge bundles over elliptically fibered Calabi-Yau three-folds. Construction of realistic theories is facilitated by the appearance of non-perturbative five-branes in the vacuum. The complete moduli space of these five-branes is computed and their worldvolume gauge theory discussed. In the context of holomorphic gauge bundles, it is shown how grand unified gauge groups can be spontaneously broken to the gauge group of the standard model. These ideas are illustrated in an explicit SU(5) three-family example.

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In seminal work, Horava and Witten [1] and Witten [2] showed that, by compactifying M-theory on an orbifold interval, $S^1/\mathbb{Z}_2$, times a Calabi–Yau three-fold, X, $\mathcal{N}=1$ supersymmetric $E_8 \times E_8$ gauge theories with chiral fermions can arise in four dimensions. In [4, 5], the effective five-dimensional heterotic M-theory was constructed and its static vacuum shown to be an exact pair of BPS three-branes located at the orbifold planes. More recently [6], this work was extended by the inclusion in the vacuum of five-branes wrapped on holomorphic curves in the Calabi–Yau three-fold and located at arbitrary points on the orbifold interval. It was emphasized that, in the context of M-theory, the standard embedding of the spin connection into the gauge connection plays no special role. Naturally, one should consider arbitrary non-standard embeddings. These vacua with five-branes and non-standard embeddings were called non-perturbative heterotic M-theory vacua.

In this letter, we give concrete form to these ideas by showing how to explicitly construct non-perturbative vacua. A more complete description will be given in [7]. First, the difficult problem of non-standard embedding is solved using techniques introduced in [8, 9] for the construction of holomorphic vector bundles over elliptically fibered Calabi–Yau three-folds. Second, the requirement that the vacua be anomaly free and $\mathcal{N}=1$ supersymmetric leads to a cohomological condition that fixes the homology class of the five-branes in terms of Chern classes of the gauge bundles and the Calabi–Yau tangent space. In this paper, we allow for the presence of five-branes in the vacua. The inclusion of five-branes greatly facilitates the solution of the cohomology condition and, hence, allows us to find many suitable vacua. We note that this construction does not require going to the M-theory limit we will consider here. It is equally applicable as a new way of building vacua with non-standard embeddings in the weakly coupled limit, though any five-brane states would have no perturbative string description.

Using our approach, we can explicitly construct non-perturbative vacua describing realistic $\mathcal{N}=1$ particle physics theories in four dimensions. The phenomenological requirements we will use are (i) that the low-energy gauge group be a suitable grand unified group, such as SU(5), SO(10) or E_6, and (ii) that there be three families. This leads us to choose the background gauge fields to lie in an SU(n) subgroup of E_8 and puts a constraint on the third Chern class of the gauge bundles. The form of this constraint for elliptically fibered Calabi–Yau three-folds was first presented in [12, 13] and some of its solutions were given in [13]. One then chooses a particular Calabi–Yau three-fold and finds the relevant SU(n) gauge bundle which is a solution of the three-family condition. That this be a true SU(n) bundle provides a second constraint. The class of the five-branes is then calculated from the cohomology condition. While the mathematics may seem involved, this procedure can be reduced to a set of rules involving properties of the base manifold of the Calabi–Yau three-fold, and a single line bundle and discrete parameter describing the bundle.

From the low-energy point of view, these theories have, in their simplest form, a visible sector with a gauge group such as SU(5), SO(10) or E_6 and a hidden sector with an unbroken E_8 gauge group. In addition, the five-branes act as a further set of hidden sectors, generically with unitary gauge groups on their worldvolumes. In five dimensions, each such hidden sector lives on a separate domain wall at a different point in the orbifold interval. We find that many realistic particle physics vacua can be constructed this way. In this paper, we consider one example, with gauge group SU(5), in detail. In particular, we give a complete description of the five-branes in this example, calculating their moduli space, the genus of the five-brane curve within the Calabi–Yau manifold and discussing the form of the low-energy theory on each five-brane worldvolume.

Having constructed grand unified theories, it is essential to show how these theories are spontaneously broken to the standard model. In this paper, we will demonstrate that this is indeed possible, within the context of holomorphic bundles over elliptically fibered Calabi–Yau three-folds, using Wilson lines. This requires constructing Calabi–Yau spaces with non-trivial fundamental group.

Finally, we note that the authors of [8] were primarily concerned with understanding the duality between heterotic theory and F-theory compactifications. In this letter, we will not discuss this duality. Here, we simply remark that the non-perturbative vacua discussed in this letter are more general than those usually considered, since the orientation of the five-branes is such that they do not, in general, map under duality to three-branes in F-theory.

We will consider vacuum states of M-theory with the following structure. The space-time structure of the vacuum is chosen to be that of Horava and Witten [1, 2] and Witten [3], to lowest order in the expansion parameter $\kappa^{2/3}$.

- Space-time is taken to have the form

$$M_{11} = M_4 \times S^1/\mathbb{Z}_2 \times X \tag{1}$$

  where M_4 is four-dimensional Minkowski space, $S^1/\mathbb{Z}_2$ is a one-dimensional orbifold and X is a smooth Calabi–Yau three-fold.

The vacuum space-time structure becomes more complicated at the next order in $\kappa^{2/3}$, but this "deformation" can be viewed as arising as the static vacuum of the five-dimensional effective theory [4, 5] and, hence, need not concern us here.

The $\mathbb{Z}_2$ orbifold projection necessitates the introduction, on each of the two orbifold planes, of an $\mathcal{N} = 1$, E_8 Yang-Mills supermultiplet which is required for anomaly cancelation. The gauge field structure of these vacua is encoded in the gauge bundle. To be compatible with four preserved supercharges in four dimensions, the gauge bundle on each orbifold plane must be a semi-stable, holomorphic bundle with the structure group being the complexification $E_{8\mathbf{C}}$ of E_8. We will denote any group

G and its complexification $G_{\mathbf{C}}$ simply as G, letting context dictate which group is being referred to. These semi-stable, holomorphic gauge bundles are, a priori, allowed to be arbitrary in all other respects. In particular, there is no requirement that the spin-connection of the Calabi–Yau three-fold be embedded into an SU(3) subgroup of the gauge connection of one of the E_8 bundles, the so-called standard embedding. This generalization to arbitrary semi-stable holomorphic gauge bundles is what is referred to as non-standard embedding. These terms are somewhat irrelevant in the context of M-theory, where no choice of embedding can ever set the entire eleven-dimensional four-form field strength to zero. For this reason, we will simply refer to arbitrary semi-stable holomorphic E_8 gauge bundles. It is clear that we can restrict the transition functions to be elements of any subgroup G of E_8, such as $G = \mathrm{SU}(n)$ or $Sp(n)$. Hence, we choose the following gauge structure for the vacua.

- There is a semi-stable holomorphic gauge bundle V_i with fiber group $G_i \subseteq E_8$ over the Calabi-Yau three-fold on the i-th orbifold fixed plane for $i = 1, 2$. The structure groups G_1 and G_2 of the two bundles can be any subgroups of E_8 and need not be the same.

In addition, as discussed in [3, 6], we can allow for the presence of five-branes located at points throughout the orbifold interval. The five-branes will preserve $\mathcal{N} = 1$ supersymmetry, provided they are wrapped on holomorphic two-cycles within X and otherwise span the flat Minkowski space M_4 [3, 10, 11]. The inclusion of five-branes is essential for a complete discussion of M-theory vacua. The reason for this is that, given a Calabi–Yau three-fold background, the presence of five-branes allows one to construct large numbers of gauge bundles that would otherwise be disallowed [6].

- We allow for the presence of five-branes in the vacuum, which are wrapped on holomorphic curves W within X.

The requirements of gauge and gravitational anomaly cancelation on the two orbifold fixed planes, as well as anomaly cancelation on each five-brane worldvolume, necessitates the addition of four-form sources to the four-form field strength Bianchi identity. Integrate this Bianchi identity over any five-cycle which spans the orbifold interval together with an arbitrary four-cycle $\mathcal{C}_4$ in the Calabi-Yau three-fold. Since dG is exact and the cycle is compact, this integral must vanish. This means the sources are subject to the following condition.

- The Calabi-Yau three-fold, the gauge bundles and the five-branes are subject to the cohomological constraint

$$c_2(V_1) + c_2(V_2) + [W] = c_2(TX)\,, \tag{2}$$

where $c_2(V_i)$ and $c_2(TX)$ are the second Chern classes of the gauge bundle V_i and the tangent bundle TX respectively and $[W]$ is the cohomology class associated with the five-brane curves W.

Note that integrating this constraint over an arbitrary four-cycle $\mathcal{C}_4$ yields the expression

$$n_1(\mathcal{C}_4) + n_2(\mathcal{C}_4) + n_5(\mathcal{C}_4) = n_R(\mathcal{C}_4)\,, \tag{3}$$

which states that the net magnetic charge on the four-cycle $\mathcal{C}_4$ must vanish, so that the sum of the number of gauge instantons on the two orbifold planes, plus the sum of the five-brane magnetic charges, must equal the instanton number for the Calabi-Yau tangent bundle, a number which is fixed once the Calabi-Yau three-fold is chosen. Vacua of this type will be referred to as non-perturbative heterotic M-theory vacua.

The discussion given thus far is completely generic, in that it applies to any Calabi-Yau three-fold and any gauge bundles that can be constructed over it. However, realistic particle physics theories require the explicit construction of these gauge bundles. Here, we will present a formalism for choosing appropriate semi-stable holomorphic gauge bundles with structure groups G_1 and G_2 over the two orbifold fixed planes. In this letter, for specificity, we will restrict the structure groups to be

$$G_i = \mathrm{SU}(n_i)\,, \tag{4}$$

for $i = 1, 2$. Other structure groups, such as $Sp(n)$ or exceptional groups, will be discussed elsewhere.

Our explicit bundle constructions [8, 9] will be achieved over the restricted, but rich, set of elliptically fibered Calabi-Yau three-folds. These three-folds are known to be the simplest Calabi-Yau spaces on which one can explicitly construct bundles, compute Chern classes, moduli spaces and so on. This makes them a compelling choice for the construction of concrete particle physics theories. Having constructed the bundles, one can explicitly calculate the gauge bundle Chern classes $c_2(V_i)$ for $i = 1, 2$, as well as the tangent bundle Chern class $c_2(TX)$. Having done so, one can then find the class $[W]$ of the five-branes using the cohomology condition (2).

Elliptically fibered Calabi–Yau vacua have the following properties.

- An elliptically fibered Calabi–Yau three-fold X is composed of a two-fold base B, elliptic curves (that is tori) E_b fibered over each point $b \in B$ and an analytic map $\pi : X \to B$. In this letter, we will assume there is a global section σ.

The elliptic fibration is characterized by a single line bundle $\mathcal{L}$ over B. The condition that the first Chern class of the tangent bundle TX of the Calabi–Yau three-fold X vanish implies that $\mathcal{L} = -K_B$, where K_B is the canonical bundle of the base B. The fact that $\mathcal{L}$ cannot be arbitrary limits the number of possible bases [14, 15].

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- If the base is smooth and preserves only $\mathcal{N} = 1$ supersymmetry in four-dimensions, then B is restricted to the following manifolds: an Enriques surface, a del Pezzo surface, a Hirzebruch surface, or a blow up of a Hirzebruch surface.
- The second Chern class of the holomorphic tangent bundle of X is given by [8]

$$c_2(TX) = c_2(B) + 11c_1(B)^2 + 12\sigma c_1(B)\,, \tag{5}$$

  where $c_1(B)$ and $c_2(B)$ are the first and second Chern classes of the tangent bundle of the base B.

Each gauge bundle over the elliptically fibered Calabi–Yau three-fold has the following properties.

- To specify a semi-stable $\mathrm{SU}(n)$ gauge bundle V, we need to fix a spectral cover and a line bundle $\mathcal{N}$ over it. The class of the spectral cover is itself specified in terms of a second line bundle $\mathcal{M}$ on the base B. The relevant quantities associated with $\mathcal{M}$ and $\mathcal{N}$ are their first Chern classes

$$\eta = c_1(\mathcal{M}) \tag{6}$$

  and $c_1(\mathcal{N})$ respectively. The class $c_1(\mathcal{N})$ can be expressed in terms of n, σ, η and $c_1(B)$, and an additional rational coefficient λ.
- The condition that $c_1(\mathcal{N})$ be an integer leads to the constraints on η and λ given by

$$n \text{ is odd}\,, \qquad \lambda = m + \frac{1}{2} \tag{7}$$

$$n \text{ is even}\,, \qquad \lambda = m, \quad \eta = c_1(B) \mod 2\,, \tag{8}$$

  where m is an integer.
- The relevant Chern classes of an $\mathrm{SU}(n)$ gauge bundle V are given by [8, 13]

$$c_1(V) = 0 \tag{9}$$

$$c_2(V) = \eta\sigma - \frac{1}{24}c_1(B)^2\left(n^3 - n\right) + \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right) n\eta\left(\eta - nc_1(B)\right) \tag{10}$$

$$c_3(V) = 2\lambda\sigma\eta\left(\eta - nc_1(B)\right) \tag{11}$$

What gauge bundles should we choose in order to construct realistic particle physics models? The simplest case is to choose an arbitrary semi-stable $\mathrm{SU}(n)$ gauge bundle V_1, which we henceforth call V, on the first orbifold plane, but always take the gauge bundle V_2 to be trivial. Physically, this corresponds to allowing observable sector gauge groups to be subgroups, such as $\mathrm{SU}(5)$, $\mathrm{SO}(10)$ or E_6, of E_8 but leaving the hidden sector E_8 gauge group unbroken. These gauge groups arise as the commutants of the SU(n) structure groups. Explicitly, we have the following maximal subgroups of E_8 together with the decompositions of the adjoint representation of E_8 under those subgroups.

$$\begin{aligned} \mathrm{SU}(3)\times E_6: &\quad \mathbf{248}=(\mathbf{8},\mathbf{1})\oplus(\mathbf{1},\mathbf{78})\oplus(\mathbf{3},\mathbf{27})\oplus(\bar{\mathbf{3}},\bar{\mathbf{27}}) \\ \mathrm{SU}(4)\times\mathrm{SO}(10): &\quad \mathbf{248}=(\mathbf{15},\mathbf{1})\oplus(\mathbf{1},\mathbf{45})\oplus(\mathbf{4},\mathbf{16})\oplus(\bar{\mathbf{4}},\bar{\mathbf{16}}) \\ \mathrm{SU}(5)\times\mathrm{SU}(5): &\quad \mathbf{248}=(\mathbf{24},\mathbf{1})\oplus(\mathbf{1},\mathbf{24})\oplus(\mathbf{10},\mathbf{5})\oplus(\bar{\mathbf{10}},\bar{\mathbf{5}})\oplus(\mathbf{5},\bar{\mathbf{10}})\oplus(\bar{\mathbf{5}},\mathbf{10}) \end{aligned} \tag{12}$$

Thus, for example, the observable sector gauge group SO(10) arises as the subgroup of E_8 commutant with the structure group SU(4). Note that choosing V_2 to be trivial is done only for simplicity. Our formalism also allows a complete analysis of the general case where the hidden sector E_8 gauge group is broken by a non-trivial bundle V_2.

With this choice of gauge bundle, it follows from equation (2) that the cohomology class associated with the five-branes is given by

$$[W]=c_2(TX)-c_2(V)\,. \tag{13}$$

Inserting the expressions (5) and (10) into this equation allows one to explicitly determine the five-brane class $[W]$. Since we are interested in the structure of the five-brane curves, we find it expedient to use Poincaré duality to rewrite cohomological expressions in terms of homology classes. In this context, we must introduce the homology class of the fiber, which is denoted by F. If our Calabi–Yau threefold is sufficiently general, we find the following.

- The homology class associated with the five-branes is specifically of the form

  $$[W]=W_B+a_fF\,, \tag{14}$$

  where W_B is the homology class in the base,

  $$W_B = 12c_1(B)-\eta \tag{15}$$
  $$a_f = c_2(B)+c_1(B)^2\left(11+\frac{1}{24}(n^3-n)\right)-\frac{n}{2}\left(\lambda^2-\frac{1}{4}\right)\eta\,(\eta-nc_1(B)) \tag{16}$$

  and $c_1(B)$ and $c_2(B)$ are the first and second Chern classes of the base B.

Each fivebrane must be wrapped on a subspace of X, in particular some holomorphic curve C_i, but can also wrap many times, so in general has some non-negative multiplicity a_i. Thus we can conclude that to describe a set of physical five-branes, $[W]$ must be the homology class of what is called an effective curve. That is there must be a representative curve in $[W]$ which is of the form

$$W=\sum_i a_iC_i\,, \tag{17}$$

with irreducible holomorphic curves C_i and integer multiplicities a_i.

- Effective condition: The requirement that $[W]$ is the class of a set of physical five-branes, constrains $[W]$ to be effective. We can show [7], for this to be the case, that one must guarantee

$$W_B \text{ is effective in } B\,, \qquad a_f \geq 0 \text{ integer} \tag{18}$$

This requirement is a strong constraint on realistic non-perturbative vacua.

Another obvious physical criterion for constructing realistic particle physics models is that we should be able to find theories with a small number of families, preferably three. We will see that this is, in fact, easy to do via the bundle constructions on elliptically fibered Calabi–Yau three-folds that we are discussing. From the fact that all the relevant fields are in the fundamental representation of fiber group $\mathrm{SU}(n)$, we have that the number of generations is

$$N_{\mathrm{gen}} = \mathrm{index}\,(\not{D}, V) = \int_X \mathrm{td}\,(X)\mathrm{ch}\,(V) = \frac{1}{2}\int_X c_3(V)\,, \tag{19}$$

where $\mathrm{td}\,(X)$ is the Todd class of X. For the case of $\mathrm{SU}(n)$ bundles on elliptically fibered Calabi–Yau three-folds, one can show, using equation (11) above and integrating over the fiber, that the number of families becomes [13]

$$N_{\mathrm{gen}} = \lambda\eta(\eta - nc_1(B))\,. \tag{20}$$

Restricting to three families and inserting equation (15) leads to the following condition on the vacua.

- Three family condition: The requirement that the theory have three families imposes the constraint that

$$3 = \lambda\left(W_B^2 - (24-n)W_B c_1(B) + 12(12-n)c_1(B)^2\right). \tag{21}$$

To these conditions, we reiterate the above constraint on λ and η rewritten, however, in terms of W_B.

- Bundle condition: The condition that $c_1(\mathcal{N})$ be an integer leads to the constraints on W_B and λ given by

$$\begin{aligned} &n \text{ is odd}\,, \qquad \lambda = m + \frac{1}{2} \\ &n \text{ is even}\,, \qquad \lambda = m\,, \qquad W_B = c_1(B) \mod 2\,, \end{aligned} \tag{22}$$

  where m is an integer.

It is important to note that all quantities and constraints have now been reduced to properties of the base two-fold B. Specifically, if we know $c_1(B)$, $c_2(B)$, as well as a basis of effective classes in B in which to expand W_B, we will be able to exactly specify all appropriate non-perturbative vacua. Hence, one proceeds as follows.

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- If we denote by $G = \mathrm{SU}(n)$ the structure group of the gauge bundle and by H its commutant in E_8, then, for example

$$\begin{aligned} G &= \mathrm{SU}(3) \Rightarrow H = E_6 \\ G &= \mathrm{SU}(4) \Rightarrow H = \mathrm{SO}(10) \\ G &= \mathrm{SU}(5) \Rightarrow H = \mathrm{SU}(5) \end{aligned} \tag{23}$$

  and H corresponds to the low energy gauge group of the theory. Choose the desired gauge group H and, hence, the structure group G.

- Choose a base B (an Enriques, a del Pezzo or a Hirzebruch surface or its blow up) for the Chern classes $c_1(B)$ and $c_2(B)$, as well as the effective classes.

- Specify W_B and λ subject to effectiveness (18), three-family (21) and bundle (22) constraints.

We can use this prescription to produce numerous examples of three family models with realistic gauge groups [7]. In this letter, we will present one example which illustrates all the major issues.

Example: $B = dP_8$**.** We begin by choosing

$$H = \mathrm{SU}(5) \tag{24}$$

as the gauge group for our model. Then it follows from (23) that we must choose the structure group of the gauge bundle to be

$$G = \mathrm{SU}(5) \tag{25}$$

and, hence, $n = 5$.

At this point, it is necessary to explicitly choose the base surface, which we take to be

$$B = dP_8\,. \tag{26}$$

For the del Pezzo surface dP_8, a basis for $H_2(dP_8, \mathbb{Z})$ composed entirely of effective classes is given by l and E_i for $i = 1, \ldots, 8$ with intersection numbers

$$l \cdot l = 1\,, \qquad l \cdot E_i = 0\,, \qquad E_i \cdot E_j = -\delta_{ij}\,. \tag{27}$$

Furthermore, the first and second Chern classes of the dP_8 are given by

$$c_1(B) = 3l - \sum_{r=1}^{8} E_i\,, \qquad c_2(B) = 11\,. \tag{28}$$

We now must specify the component of the five-brane class in the base W_B and the coefficient λ, such that the constraints (18), (21) and (22) are satisfied. Since n is odd, equation (22) tells us that $\lambda = m + 1/2$ for integer m. In this example we will choose $m = 1$ and W_B so that

$$W_B = 2E_1 + E_2 + E_3 \tag{29}$$

$$\lambda = \frac{3}{2}\,. \tag{30}$$

Since E_1, E_2 and E_3 are effective, it follows that W_B is also effective, as it must be. Using the above intersection rules, one can easily show that

$$W_B^2 = -6\,, \qquad W_B c_1(B) = 4\,, \qquad c_1(B)^2 = 1\,. \tag{31}$$

Using these results, as well as $n = 5$ and $\lambda = 3/2$, one can show

$$a_f = c_2(B) + c_1(B)^2\left(11 + \frac{1}{24}(n^3 - n)\right) - \frac{3n}{2\lambda}\left(\lambda^2 - \frac{1}{4}\right) = 17\,. \tag{32}$$

Since this is a positive integer, it follows from (18) that the full five-brane curve $[W]$ is effective, as it must be. Finally, we see that

$$\lambda\left(W_B^2 - (24 - n)W_B c_1(B) + 12(12 - n)c_1(B)^2\right) = 3 \tag{33}$$

and, therefore, the three family condition (21) is satisfied.

This completes our construction of this explicit non-perturbative vacuum. It represents a model of particle physics with three families and gauge group $H = \mathrm{SU}(5)$, along with explicit five-branes wrapped on a holomorphic curve specified by

$$[W] = 2E_1 + E_2 + E_3 + 17F\,. \tag{34}$$

We will now explore the properties of the five-branes and their moduli space in detail.

The inclusion of five-branes not only generalizes the types of bundles one can consider, but also introduces new degrees of freedom into the theory, namely, the dynamical fields on the five-branes themselves. From the point of view of a five-dimensional effective theory on $M_4 \times S^1/\mathbb{Z}_2$, since two of the five-brane directions are compactified, it appears as a flat three-brane located at some point X^{11} on the orbifold. We have shown in previous work [6] that the low-energy four-dimensional theory describing the five-brane dynamics will have $\mathcal{N} = 1$ supersymmetry. Furthermore, in general, the low-energy theory of a single five-brane wrapped on a genus g holomorphic curve W has gauge group $\mathrm{U}(1)^g$ with g $\mathrm{U}(1)$ vector multiplets and a universal chiral multiplet with bosonic fields (a, X^{11}). The gauge fields and axionic scalar field a arise from the dimensional reduction of the self-dual three-form h on the five-brane worldvolume. In additional, there are chiral multiplets describing the moduli space of the curve W in the Calabi–Yau three-fold.

This description of the low-energy five-brane is correct for a generic five-brane. However, it is important to note that the gauge group can be enhanced, either when two or more five-branes overlap or when a single five-brane degenerates in such a way that two parts of the curve W come close together in the Calabi–Yau manifold. In the following analysis of the five-brane moduli space, we will see examples of both types of enhancement. As an example, for n overlapping five-branes, each wrapped on the same genus g curve W, the n copies of $\mathrm{U}(1)^g$ are enhanced to $\mathrm{U}(n)^g$.

Within the context of the explicit three family, $H = \mathrm{SU}(5)$ vacuum discussed above, having so far only fixed the homology class $[W]$, let us now analyze the moduli space of the curve W in the Calabi–Yau three-fold, as well as discuss the five-brane location moduli X^{11}. We can construct the complete moduli space for this theory. However, in this letter, we will present only a part of this space which is sufficient to illustrate our main points. In general, there are different branches of the moduli space corresponding to different numbers of five-brane. The number of five-branes, and their physical properties, will emerge from our analysis. In particular, we will be able to exactly calculate the genus of the five-brane curves and hence the resulting low-energy gauge group.

The five-brane class $[W]$ in the above example is given by

$$[W] = 2E_1 + E_2 + E_3 + 17F\,. \tag{35}$$

We see that we have 17 copies of the fiber together with two copies of E_1 and a single copy each of E_2 and E_3 in the base. Now, some of the F part of the homology class can be shared among the three distinct E_i components. We are led to the following generic decomposition

$$\begin{aligned} [W] &= [W_1] + [W_2] + [W_3] + [W_4] \\ &= (2E_1 + aF) + (E_2 + bF) + (E_3 + cF) + (17 - a - b - c)\,F\,, \end{aligned} \tag{36}$$

where a, b, and c are integers. By this decomposition, we mean that there are at least four separate five-branes (at least four because each curve W_i can further decompose into several five-branes), except in the case where $17 - a - b - c = 0$ when there are at least three. Physically this means that these five- branes lie at arbitrary positions X^{11} along the orbifold interval $I = S^1/\mathbb{Z}_2$. Note that for each of the four components to be effective, we must restrict a, b, c and $17 - a - b - c$ to each be non-negative, which we do henceforth. There are clearly many different ways we can distribute F among the three curves. Each distribution corresponds to a different component of moduli space. We conclude that the five-brane class in this specific example has a large moduli space specified by the restricted set of integers a, b, and c. In order to determine the exact number of five-branes and their physical properties, we must examine the individual moduli spaces for each of these four generic components.

Let us first consider the curve

$$[W_4] = (17 - a - b - c)\,F \equiv fF\,. \tag{37}$$

W_4	number of components	genus	moduli space
fF	1	1	$dP_8 \times I$
$f_1F + f_2F$	2	$1+1$	$(dP_8 \times I)^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$F + \cdots + F$	f	$f \times 1$	$(dP_8 \times I)^f$

Table 1: Complete moduli space for the curve W_4.

The complete moduli space for the curve W_4 (ignoring the axionic scalar field a), for a fixed value of $f = 17 - a - b - c$, is given in table 1, where $f_1 + \cdots + f_n = f$ and we have not included the fact that if there is a repetition of some of the f_i, we may have to mod out the moduli space by a discrete group.

- Example: Consider the second line of the table. This represents two independent five-branes lying at arbitrary positions X^{11} along the orbifold interval I. Each five-brane is multiply wrapped on a holomorphic curve in X with moduli space dP_8. Since the genus of each holomorphic curve is unity, one might expect a single U(1) gauge field on each five-brane worldvolume. However, the multiple wrapping means that the five-branes are overlapping themselves. The result is that the gauge group on one brane is enhanced to $\mathrm{U}(f_1)$ whereas the other brane enhances to $\mathrm{U}(f_2)$. This is an example of the second type of enhancement mentioned above. Since the locations of the two branes are arbitrary, there is a region of moduli space where they approach each other and overlap. These overlapping branes further enhance the gauge group from $\mathrm{U}(f_1) \times \mathrm{U}(f_2)$ to $\mathrm{U}(f)$ where $f = f_1 + f_2$. This is an example of the first type of enhancement. As in the case of D-branes, the whole moduli space of W_4 can be viewed as the moduli space of vacua of a four-dimensional $\mathrm{U}(f)$ theory. The different examples listed above correspond to different Higgs branches of the theory.

We now consider the W_2 and W_3 components. The construction of the moduli space of W_3 proceeds in an identical fashion to that of W_2. Hence, we will limit the discussion here to the class $[W_2]$. This is given by

$$[W_2] = E_2 + bF\,. \tag{38}$$

As a concrete example, we will take $b = 2$. However, the analysis would be similar for any positive integer b. One can show that the curve W_2 must lie in a dP_9 surface within the Calabi–Yau threefold. From our knowledge of del Pezzo surfaces, we recall that a basis of curves on dP_9 are l' and E'_i for $i = 1, \ldots, 9$, where the prime distinguishes these curves from their counterparts in the base $B = dP_8$ introduced above. A subset of this moduli space is shown in table 2.

W_2	number of components	genus	moduli space
$l' - E_1' - E_2'$	1	0	I
$2l' - E_1' - E_2' - E_3' - E_4' - E_5'$	1	0	I
$3l - \sum_{j=2}^{9} E_j'$	1	1	$\mathbb{P}^1 \times I$
$E_2' + 2\left(3l - \sum_{j=1}^{9} E_j'\right)$	3	$1+0+1$	$\mathbb{P}^2 \times I^3$

Table 2: A subset of the moduli space for the curve W_2.

- Example: Consider the first line of this table. This represents one five-brane lying at some position X^{11} along the orbifold interval I. This five-brane is wrapped once around a holomorphic curve in X with a moduli space consisting of a single isolated point with no freedom to deform within the Calabi–Yau space. Since the genus of the holomorphic curve is $g = 0$, it follows that this brane has no gauge group on its worldvolume. We want to emphasize this situation. By choosing the W_2 curve to lie at such an isolated point of moduli space, one can dramatically reduce the number of such moduli in the theory.

Let us now briefly discuss the remaining component

$$[W_1] = 2E_2 + aF\,. \tag{39}$$

For specificity, we consider the case $a = 2$, although the analysis would proceed in a similar way for any non-negative value of a. Part of the moduli space for this case is given in table 3, where $\mathcal{C}$ is the curve of zeros of the discriminant of the cubic describing the curve. Note that within each divided set of rows, a given curve in dP_9 decomposes into components. For example, the second row describes a single five-brane which, at any one of eight points in $\mathbb{P}^1$, can split into the two independent five-branes described in line three. Of particular interest is the last case, where the curve is degenerating. This will be described in more detail in [7].

W_1	number of comp.	genus	moduli space
$2E_2'$	2	$0+0$	$I \times I$
$l' - E_1'$	1	0	$\{\mathbb{P}^1 - 8 \text{ pts.}\} \times I$
$(l' - E_1' - E_2') + E_2'$	2	$0+0$	$I \times I$
$3l' - \sum_{j=3}^{9} E_j'$	1	1	$\{\mathbb{P}^2 - \mathcal{C}\} \times I$
$3l' - \sum_{j=3}^{9} E_j'$	1	0	$\{\mathcal{C} - 21 \text{ pts.}\} \times I$
$(l' - E_3' - E_4') + (2l' - E_5' - \cdots - E_9')$	2	$0+0$	$I \times I$

Table 3: Part of the moduli space for the curve W_1.

We now turn to another physically important issue. That is, in the non-perturbative vacua we are studying, how are the grand unified gauge groups, such as SU(5), SO(10) and E_6, spontaneously broken to the gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ of the standard model? From the decompositions of the E_8 gauginos given in (13), we see that the fermions in the adjoint representation of the grand unified gauge group H are singlets under the structure group $G = \mathrm{SU}(n)$. Thus, the low-energy zero-modes for these fermions are all gauginos in the grand unified gauge supermultiplet and, hence, there are no matter supermultiplets in the adjoint representation. Therefore, there are no Higgs supermultiplets to break the grand unified group H to the standard model gauge group. To achieve this symmetry breakdown, one must use Wilson lines, which require the Calabi–Yau three-fold X to have a non-trivial fundamental group. Does this occur for the non-perturbative vacua discussed in this letter?

It is most straightforward to discuss this issue within the context of a specific non-perturbative vacuum, though we note examples for other base manifolds can be constructed, in particular by relaxing the condition that the fibration has a section [7]. Let us consider an elliptically fibered Calabi-Yau three-fold with the base

$$B = K3/i \tag{40}$$

where i is the Enriques involution on $K3$ which takes the volume form to minus itself. This base B is an example of an Enriques surface. The elliptically fibered Calabi-Yau three-fold with this base has the structure

$$X = (K3 \times E)/(i, -1) \tag{41}$$

where E is an elliptic curve and the action of -1 on E is inversion in the origin. Using the methods of this letter, one can again construct bundles with fiber group $G = \mathrm{SU}(n)$ and realistic gauge groups H. Recall that H is the subgroup of E_8 that commutes with $\mathrm{SU}(n)$. Now, one can show that the first homotopy group $\pi_1(X)$ is the non-Abelian group generated by the actions $(x,y) \to (x+1,y)$, $(x,y) \to (x,y+1)$ and $(x,y) \to (-x,-y)$ in the real plane. The group isomorphism

$$\pi_1/(\mathbb{Z} \times \mathbb{Z}) \cong \mathbb{Z}_2 \tag{42}$$

then implies that the structure group of the gauge bundle can be extended from $G = \mathrm{SU}(n)$ to $G = \mathrm{SU}(n) \times \mathbb{Z}_2$. The low energy gauge group is now the subgroup $\mathcal{H} \subset E_8$ which commutes with $G = \mathrm{SU}(n) \times \mathbb{Z}_2$. Clearly $\mathcal{H}$ is the subgroup of H that commutes with $\mathbb{Z}_2$ and we can think of the $\mathbb{Z}_2$ part of the structure group as spontaneously breaking

$$H \longrightarrow \mathcal{H} \tag{43}$$

For example, first choose the structure group $G = \mathrm{SU}(5)$. It follows that H is the grand unified group $H = \mathrm{SU}(5)$. Now extend the bundle so that $G = \mathrm{SU}(n) \times \mathbb{Z}_2$.

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The embedding of $\mathbb{Z}_2$ in $H = \mathrm{SU}(5)$ can be chosen so that

$$\mathcal{G} = \begin{pmatrix} \mathbf{1}_3 & \\ & -\mathbf{1}_2 \end{pmatrix} \tag{44}$$

where $\mathcal{G}$ is the generating element of $\mathbb{Z}_2$. It follows that the $\mathbb{Z}_2$ part of the fiber bundle spontaneously breaks

$$\mathrm{SU}(5) \longrightarrow \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \tag{45}$$

and the standard model gauge group structure is achieved.

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